

# SOME SPECIAL CASES OF BOBADILLA'S CONJECTURE

BRIAN HEPLER AND DAVID B. MASSEY

**ABSTRACT.** We prove two special cases of a conjecture of J. Fernández de Bobadilla for hypersurfaces with 1-dimensional critical loci.

We do this via a new numerical invariant for such hypersurfaces, called the beta invariant, first defined and explored by the second author in 2014. The beta invariant is an algebraically calculable invariant of the local ambient topological-type of the hypersurface, and the vanishing of the beta invariant is equivalent to the hypotheses of Bobadilla's conjecture.

## 1. INTRODUCTION

Throughout this paper, we shall suppose that  $\mathcal{U}$  is an open neighborhood of the origin in  $\mathbb{C}^{n+1}$ , and that  $f : (\mathcal{U}, \mathbf{0}) \rightarrow (\mathbb{C}, 0)$  is a complex analytic function with a 1-dimensional critical locus at the origin, i.e.,  $\dim_{\mathbf{0}} \Sigma f = 1$ . We use coordinates  $\mathbf{z} := (z_0, \dots, z_n)$  on  $\mathcal{U}$ .

We assume that  $z_0$  is generic enough so that  $\dim_{\mathbf{0}} \Sigma(f|_{V(z_0)}) = 0$ . One implication of this is that

$$V\left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \dots, \frac{\partial f}{\partial z_n}\right)$$

is purely 1-dimensional at the origin. As analytic cycles, we write

$$\left[V\left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \dots, \frac{\partial f}{\partial z_n}\right)\right] = \Gamma_{f,z_0}^1 + \Lambda_{f,z_0}^1,$$

where  $\Gamma_{f,z_0}^1$  and  $\Lambda_{f,z_0}^1$  are, respectively, the relative polar curve and 1-dimensional Lê cycle; see [7] or the section.

We recall a classical non-splitting result (presented in a convenient form here) proved independently by Gabrielov, Lazzeri, and Lê (in [3], [4], and [5], respectively) regarding the non-splitting of the cohomology of the Milnor fiber of  $f|_{V(z_0)}$  over the critical points of  $f$  in a nearby hyperplane slice  $V(z_0 - t)$  for a small non-zero value of  $t$ .

**Theorem 1.1** (GLL non-splitting). *The following are equivalent:*

- (1) *The Milnor number of  $f|_{V(z_0)}$  at the origin is equal to*

$$\sum_C \mu_C^\circ (C \cdot V(z_0))_{\mathbf{0}},$$

*where the sum is over the irreducible components  $C$  of  $\Sigma f$  at  $\mathbf{0}$ ,  $(C \cdot V(z_0))_{\mathbf{0}}$  denotes the intersection number of  $C$  and  $V(z_0)$  at  $\mathbf{0}$ , and  $\mu_C^\circ$  denotes the Milnor number of  $f$ , restricted to a generic hyperplane slice, at a point  $\mathbf{p} \in C \setminus \{\mathbf{0}\}$  close to  $\mathbf{0}$ .*

- (2)  *$\Gamma_{f,z_0}^1$  is zero at the origin (i.e.,  $\mathbf{0}$  is not in the relative polar curve).*

*Furthermore, when these equivalent conditions hold,  $\Sigma f$  has a single irreducible component which is smooth and is transversely intersected by  $V(z_0)$  at the origin.*

---

2010 *Mathematics Subject Classification.* 32B15, 32C18, 32B10, 32S25, 32S15, 32S55.

*Key words and phrases.* Lê's conjecture, Bobadilla's conjecture, Milnor fiber, 1-dimensional critical locus, hypersurface.

This paper is concerned with a recent conjecture made by Javier Fernández de Bobadilla, positing that, in the spirit of Theorem 1.1, the cohomology of the Milnor fiber of  $f$ , **not of a hyperplane slice**, does not split. We state a slightly more general form of Bobadilla's original conjecture, for the case where  $\Sigma f$  may, a priori, have more than a single irreducible component:

**Conjecture 1.2** (Fernández de Bobadilla). *Denote by  $F_{f,0}$  the Milnor fiber of  $f$  at the origin. Suppose that  $\tilde{H}^*(F_{f,0}; \mathbb{Z})$  is non-zero only in degree  $(n-1)$ , and that*

$$\tilde{H}^{n-1}(F_{f,0}; \mathbb{Z}) \cong \bigoplus_C \mathbb{Z}^{\mu_C^\circ}$$

*where the sum is over all irreducible components  $C$  of  $\Sigma f$  at  $0$ . Then, in fact,  $\Sigma f$  has a single irreducible component, which is smooth.*

Bobadilla's conjecture, in its original phrasing ([1]), is a reformulation of a conjecture of Lê (see, for example, [2]): if  $(X, 0)$  is a reduced surface germ in  $(\mathbb{C}^3, 0)$ , and the (real) link of  $X$  is homeomorphic to a sphere, then  $X$  is (analytically) isomorphic to the total space of an equisingular deformation of an irreducible plane curve.

We approach Conjecture 1.2 via the **beta invariant** of a hypersurface with a 1-dimensional critical locus, first defined and explored by the second author in [6]. The beta invariant,  $\beta_f$ , of  $f$  is an invariant of the local ambient topological-type of the hypersurface  $V(f)$ . It is a non-negative integer, and is algebraically calculable.

Our motivation for using this invariant is that the requirement that  $\beta_f = 0$  is precisely equivalent to the hypotheses of Conjecture 1.2, essentially turning the problem into a purely algebraic question [see 6, Theorem 5.4] For this reason, we will refer to our new formulation of 1.2 as the **Beta Conjecture**.

In this paper, we give proofs of the **Beta Conjecture** in two special cases:

- (1) In Corollary 3.2, we prove an induction-like result for when  $f$  is a sum of two analytic functions defined on disjoint sets of variables.
- (2) In Theorem 4.2, we prove the result for the case when the relative polar curve  $\Gamma_{f,z_0}^1$  is defined by a single equation inside the relative polar surface  $\Gamma_{f,z}^2$  (see below).

## 2. NOTATION AND KNOWN RESULTS

The bulk of this section is largely a summary of the concepts of Chapter 1 of [7], which will be used throughout this paper.

Our assumption that  $\dim_0 \Sigma(f|_{V(z_0)}) = 0$  is equivalent to assuming that the variety  $V\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$  is purely 1-dimensional (and non-empty) at  $0$  and is intersected properly by the hyperplane  $V(z_0)$  at  $0$ .

**Definition 2.1.** *The **relative polar surface of  $f$  with respect to  $z$** , denoted  $\Gamma_{f,z}^2$ , is, as an analytic cycle at the origin,  $\left[V\left(\frac{\partial f}{\partial z_2}, \dots, \frac{\partial f}{\partial z_n}\right)\right]$ . Note that each component of this at the origin must be precisely 2-dimensional, and so is certainly not contained in  $\Sigma f$ .*

*The **relative polar curve of  $f$  with respect to  $z_0$** , denoted  $\Gamma_{f,z_0}^1$ , is, as an analytic cycle at the origin, the collection of those components of the cycle  $\left[V\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)\right]$  which are not contained in  $\Sigma f$ .*

*The **1-dimensional Lê cycle of  $f$  with respect to  $z_0$** , at the origin, denoted  $\Lambda_{f,z_0}^1$ , consists of those components of  $\left[V\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)\right]$  at the origin which are contained in  $\Sigma f$ .*

We sometimes enclose an analytic variety  $V$  in brackets to indicate that we are considering  $V$  as a cycle. We do, however, frequently omit this notation if it is clear from context that a given variety is to be considered as an analytic cycle.

An immediate consequence of Definition 2.1 is that, as cycles on  $\mathcal{U}$ ,

$$V\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right) = \Gamma_{f,z_0}^1 + \Lambda_{f,z_0}^1.$$

We will use this identity throughout this paper.

Note that, by assumption,  $V\left(\frac{\partial f}{\partial z_0}\right)$  properly intersects  $\Gamma_{f,z_0}^1$  at  $\mathbf{0}$ , and also that  $V(z_0)$  properly intersects  $\Lambda_{f,z_0}^1$  at  $\mathbf{0}$ .

Letting  $C$ 's denote the underlying reduced components of  $\Sigma f$  at  $\mathbf{0}$ , we have (as cycles at the origin)

$$\Lambda_{f,z_0}^1 = \sum_C \mu_C^\circ [C],$$

where  $\mu_C^\circ$  denotes the Milnor number of  $f$ , restricted to a generic hyperplane slice, at a point  $\mathbf{p} \in C \setminus \{\mathbf{0}\}$  close to  $\mathbf{0}$  ([see 7, Remark 1.19]).

**Definition 2.2.** *The intersection numbers  $\left(\Gamma_{f,z_0}^1 \cdot V\left(\frac{\partial f}{\partial z_0}\right)\right)_{\mathbf{0}}$  and  $\left(\Lambda_{f,z_0}^1 \cdot V(z_0)\right)_{\mathbf{0}}$  are, respectively, the **Lê numbers**  $\lambda_{f,z_0}^0$  and  $\lambda_{f,z_0}^1$  (at the origin).*

*Via the above formula for  $\Lambda_{f,z_0}^1$ , we have:*

$$\lambda_{f,z_0}^1 = \sum_C \mu_C^\circ (C \cdot V(z_0))_{\mathbf{0}}.$$

A fundamental property of Lê numbers from [7] is:

**Proposition 2.3.** *Let  $\tilde{b}_n(F_{f,\mathbf{0}})$  and  $\tilde{b}_{n-1}(F_{f,\mathbf{0}})$  denote the reduced Betti numbers of the Milnor fiber of  $f$  at the origin. Then,*

$$\tilde{b}_n(F_{f,\mathbf{0}}) - \tilde{b}_{n-1}(F_{f,\mathbf{0}}) = \lambda_{f,z_0}^0 - \lambda_{f,z_0}^1.$$

We will need the following classical relations between intersection numbers.

**Proposition 2.4.** *Since  $\dim_{\mathbf{0}} \Sigma(f|_{V(z_0)}) = 0$ :*

- (1)  $\dim_{\mathbf{0}} \Gamma_{f,z_0}^1 \cap V(f) \leq 0$ ,  $\dim_{\mathbf{0}} \Gamma_{f,z_0}^1 \cap V(z_0) \leq 0$ ,  $\dim_{\mathbf{0}} \Gamma_{f,z_0}^1 \cap V\left(\frac{\partial f}{\partial z_0}\right) \leq 0$ , and

$$(\Gamma_{f,z_0}^1 \cdot V(f))_{\mathbf{0}} = (\Gamma_{f,z_0}^1 \cdot V(z_0))_{\mathbf{0}} + \left(\Gamma_{f,z_0}^1 \cdot V\left(\frac{\partial f}{\partial z_0}\right)\right)_{\mathbf{0}}.$$

*The proof of this result is sometimes referred to as Teissier's trick.*

- (2) *In addition,*

$$\mu_{\mathbf{0}}(f|_{V(z_0)}) = (\Gamma_{f,z_0}^1 \cdot V(z_0))_{\mathbf{0}} + (\Lambda_{f,z_0}^1 \cdot V(z_0))_{\mathbf{0}}.$$

Formula (1) above was first proved by B. Teissier in [14] for functions with isolated critical points, and it is an easy exercise to show that the result still holds in the case where  $f$  has a critical locus of arbitrary dimension. Formula (2) follows from the fact that

$$\Sigma(f|_{V(z_0)}) = V\left(z_0, \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$$

and the fact that  $V(z_0)$  properly intersects  $V\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$  at the origin.

The following numerical invariant, defined and discussed in [6], is crucial to the contents and goal of this paper.

**Definition 2.5.** *The **beta invariant** of  $f$  with respect to  $z_0$  is:*

$$\begin{aligned}\beta_f = \beta_{f,z_0} &:= \left( \Gamma_{f,z_0}^1 \cdot V \left( \frac{\partial f}{\partial z_0} \right) \right)_0 - \sum_C \mu_C^\circ [(C \cdot V(z_0))_0 - 1] \\ &= \lambda_{f,z_0}^0 - \lambda_{f,z_0}^1 + \sum_C \mu_C^\circ \\ &= \tilde{b}_n(F_{f,0}) - \tilde{b}_{n-1}(F_{f,0}) + \sum_C \mu_C^\circ.\end{aligned}$$

Using Proposition 2.4,  $\beta_f$  may be equivalently expressed as

$$\beta_f = (\Gamma_{f,z_0}^1 \cdot V(f))_0 - \mu_0(f|_{V(z_0)}) + \sum_C \mu_C^\circ.$$

**Remark 2.6.** A key property of the beta invariant is that the value  $\beta_f$  is independent of the choice of linear form  $z_0$  (provided, of course, that the linear form satisfies  $\dim_{\mathbf{0}} \Sigma(f|_{V(z_0)}) = 0$ ). This often allows a great deal of freedom in calculating  $\beta_f$  for a given  $f$ , as different choices of linear forms  $L = z_0$  may result in simpler expressions for the intersection numbers  $\lambda_{f,z_0}^0$  and  $\lambda_{f,z_0}^1$ , while leaving the value of  $\beta_f$  unchanged. [See 6, Remark 3.2, Example 3.4].

It is shown in [6] that  $\beta_f \geq 0$ . The interesting question is how strong the requirement that  $\beta_f = 0$  is.

**Conjecture 2.7** (Beta Conjecture). *If  $\beta_f = 0$ , then  $\Sigma f$  has a single irreducible component at  $\mathbf{0}$ , which is smooth.*

**Conjecture 2.8** (polar form of the Beta Conjecture). *If  $\beta_f = 0$ , then  $\mathbf{0}$  is not in the relative polar curve  $\Gamma_{f,z_0}^1$  (i.e., the relative polar curve is 0 as a cycle at the origin).*

*Equivalently, if the relative polar curve at the origin is not empty, then  $\beta_f > 0$ .*

**Proposition 2.9.** *The [Beta Conjecture](#) is equivalent to the [polar form of the Beta Conjecture](#).*

*Proof.* Suppose throughout that  $\beta_f = 0$ .

Suppose first that the Beta Conjecture holds, so that  $\Sigma f$  has a single irreducible component at  $\mathbf{0}$ , which is smooth. Then  $\beta_f = \lambda_{f,z_0}^0 = 0$ , and so the relative polar curve must be zero at the origin.

Suppose now that the polar form of the Beta Conjecture holds, so that  $\Gamma_{f,z_0}^1 = 0$  at  $\mathbf{0}$ . Then [GLL non-splitting](#) implies that  $\Sigma f$  has a single irreducible component at  $\mathbf{0}$ , which is smooth.  $\square$

## 3. GENERALIZED SUSPENSION

Suppose that  $\mathcal{U}$  and  $\mathcal{W}$  are open neighborhoods of the origin in  $\mathbb{C}^{n+1}$  and  $\mathbb{C}^{m+1}$ , respectively, and let  $g : (\mathcal{U}, \mathbf{0}) \rightarrow (\mathbb{C}, 0)$  and  $h : (\mathcal{W}, \mathbf{0}) \rightarrow (\mathbb{C}, 0)$  be two complex analytic functions. Let  $\pi_1 : \mathcal{U} \times \mathcal{W} \rightarrow \mathcal{U}$  and  $\pi_2 : \mathcal{U} \times \mathcal{W} \rightarrow \mathcal{W}$  be the natural projection maps, and set  $f = g \boxplus h := g \circ \pi_1 + h \circ \pi_2$ . Then, one trivially has

$$\Sigma f = (\Sigma g \times \mathbb{C}^{m+1}) \cap (\mathbb{C}^{n+1} \times \Sigma h).$$

Consequently, if we assume that  $g$  has a one-dimensional critical locus at the origin, and that  $h$  has an isolated critical point at  $\mathbf{0}$ , then  $\Sigma f = \Sigma g \times \{\mathbf{0}\}$  is 1-dimensional and (analytically) isomorphic to  $\Sigma g$ .

From this, one immediately has the following result.

**Proposition 3.1.** *Suppose that  $g$  and  $h$  are as above, so that  $f = g \boxplus h$  has a one-dimensional critical locus at the origin in  $\mathbb{C}^{n+m+2}$ . Then,  $\beta_f = \mu_{\mathbf{0}}(h)\beta_g$ .*

*Proof.* This is a consequence of the Sebastiani-Thom isomorphism (see the results of Némethi [9],[10], Oka [11], Sakamoto [12], Sebastiani-Thom [13], and Massey [8]) for the reduced integral cohomology of the Milnor fiber of  $f = g \boxplus h$  at  $\mathbf{0}$ . Letting  $\widehat{C}$  denote the component of the critical locus  $f$  which corresponds to  $C$ , the Sebastiani-Thom Theorem tells us that

$$\widetilde{b}_{n+m+1}(F_{f,\mathbf{0}}) = \mu_{\mathbf{0}}(h)\widetilde{b}_n(F_{g,\mathbf{0}}), \quad \widetilde{b}_{n+m}(F_{f,\mathbf{0}}) = \mu_{\mathbf{0}}(h)\widetilde{b}_{n-1}(F_{g,\mathbf{0}}), \quad \text{and} \quad \mu_{\widehat{C}}^{\circ} = \mu_{\mathbf{0}}(h)\mu_C^{\circ}.$$

Thus,

$$\beta_f = \lambda_{f,z_0}^0 - \lambda_{f,z_0}^1 + \sum_{\widehat{C}} \mu_{\widehat{C}}^{\circ} = \widetilde{b}_{n+m+1}(F_{f,\mathbf{0}}) - \widetilde{b}_{n+m}(F_{f,\mathbf{0}}) + \sum_{\widehat{C}} \mu_{\widehat{C}}^{\circ} = \mu_{\mathbf{0}}(h)\beta_g.$$

□

**Corollary 3.2.** *Suppose  $f = g \boxplus h$ , where  $g$  and  $h$  are as in Proposition 3.1. Then, if the [Beta Conjecture](#) is true for  $g$ , it is true for  $f$ .*

*Proof.* Suppose that  $\beta_f = 0$ . By Proposition 3.1, this is equivalent to  $\beta_g = 0$ , since  $\mu_{\mathbf{0}}(h) > 0$ . By assumption,  $\beta_g = 0$  implies that  $\Sigma g$  is smooth at zero. Since  $\Sigma f = \Sigma g \times \{\mathbf{0}\}$ , it follows that  $\Sigma f$  is also smooth at  $\mathbf{0}$ , i.e., the [Beta Conjecture](#) is true for  $f$ . □

4.  $\Gamma_{f,z_0}^1$  AS A HYPERSURFACE IN  $\Gamma_{f,z}^2$ 

Let  $I := \langle \frac{\partial f}{\partial z_2}, \dots, \frac{\partial f}{\partial z_n} \rangle \subseteq \mathcal{O}_{\mathcal{U},\mathbf{0}}$ , so that the relative polar surface of  $f$  with respect to the coordinates  $\mathbf{z}$  is (as a cycle at  $\mathbf{0}$ ) given by  $\Gamma_{f,z}^2 = [V(I)]$ .

For the remainder of this section, we will drop the brackets around cycles for convenience, and assume that everything is considered as a cycle unless otherwise specified. We remind the reader that we are assuming that  $f|_{V(z_0)}$  has an isolated critical point at the origin.

**Proposition 4.1.** *The following are equivalent:*

- (1)  $\dim_{\mathbf{0}} \left( \Gamma_{f,z}^2 \cap V(f) \cap V(z_0) \right) = 0$ .
- (2) *For all irreducible components  $C$  at the origin of the analytic set  $\Gamma_{f,z}^2 \cap V(f)$ ,  $C$  is purely 1-dimensional and properly intersected by  $V(z_0)$  at the origin.*
- (3)  $\Gamma_{f,z}^2$  *is properly intersected by  $V(z_0, z_1)$  at the origin.*

Furthermore, when these equivalent conditions hold

$$(\Gamma_{f,z}^2 \cdot V(f) \cdot V(z_0))_{\mathbf{0}} = \mu_{\mathbf{0}} \left( f|_{V(z_0)} \right) + (\Gamma_{f,z}^2 \cdot V(z_0, z_1))_{\mathbf{0}}.$$

*Proof.* Clearly (1) and (2) are equivalent. We wish to show that (1) and (3) are equivalent. This follows from Tessier's trick applied to  $f|_{V(z_0)}$ , but – as it is crucial – we shall quickly run through the argument.

Since  $f|_{V(z_0)}$  has an isolated critical point at the origin,

$$\dim_{\mathbf{0}} \left( \Gamma_{f,\mathbf{z}}^2 \cap V \left( \frac{\partial f}{\partial z_1} \right) \cap V(z_0) \right) = 0.$$

Hence,  $Z := \Gamma_{f,\mathbf{z}}^2 \cap V(z_0)$  is purely 1-dimensional at the origin.

Let  $Y$  be an irreducible component of  $Z$  through the origin, and let  $\alpha(t)$  be a parametrization of  $Y$  such that  $\alpha(0) = \mathbf{0}$ . Let  $z_1(t)$  denote the  $z_1$  component of  $\alpha(t)$ . Then,

$$(f(\alpha(t)))' = \frac{\partial f}{\partial z_1} \Big|_{\alpha(t)} \cdot z_1'(t). \quad (\dagger)$$

Since  $\dim_{\mathbf{0}} Y \cap V \left( \frac{\partial f}{\partial z_1} \right) = 0$ , we conclude that  $(f(\alpha(t)))' \equiv 0$  if and only if  $z_1'(t) \equiv 0$ , which tells us that  $f(\alpha(t)) \equiv 0$  if and only if  $z_1(t) \equiv 0$ . Thus,  $\dim_{\mathbf{0}} Y \cap V(f) = 0$  if and only if  $\dim_{\mathbf{0}} Y \cap V(z_1) = 0$ , i.e., (1) and (3) are equivalent. The equality now follows at once by considering the  $t$ -multiplicity of both sides of  $(\dagger)$ .  $\square$

**Theorem 4.2.** *Suppose that*

- (1) *for all irreducible components  $C$  at the origin of the analytic set  $\Gamma_{f,\mathbf{z}}^2 \cap V(f)$ ,  $C$  is purely 1-dimensional, properly intersected by  $V(z_0)$  at the origin, and  $(C \cdot V(z_0))_{\mathbf{0}} = \text{mult}_{\mathbf{0}} C$ , and*
- (2) *the cycle  $\Gamma_{f,z_0}^1$  equals  $\Gamma_{f,\mathbf{z}}^2 \cdot V(h)$  for some  $h \in \mathcal{O}_{\mathcal{U},\mathbf{0}}$  (in particular, the relative polar curve at the origin is non-empty).*

*Then,*

$$\tilde{b}_n(F_{f,\mathbf{0}}) - \tilde{b}_{n-1}(F_{f,\mathbf{0}}) \geq (\Gamma_{f,\mathbf{z}}^2 \cdot V(z_0, z_1))_{\mathbf{0}}$$

*and so*

$$\beta_f \geq (\Gamma_{f,\mathbf{z}}^2 \cdot V(z_0, z_1))_{\mathbf{0}} + \sum_C \mu_C^\circ.$$

*In particular, the [Beta Conjecture](#) is true for  $f$ .*

*Proof.* By Proposition 4.1,

$$(\Gamma_{f,\mathbf{z}}^2 \cdot V(f) \cdot V(z_0))_{\mathbf{0}} = \mu_{\mathbf{0}} \left( f|_{V(z_0)} \right) + (\Gamma_{f,\mathbf{z}}^2 \cdot V(z_0, z_1))_{\mathbf{0}}.$$

By assumption,  $\Gamma_{f,z_0}^1 = \Gamma_{f,\mathbf{z}}^2 \cdot V(h)$ , for some  $h \in \mathcal{O}_{\mathcal{U},\mathbf{0}}$ . Then, via Proposition 2.4 and the above paragraph, we have

$$\begin{aligned} \tilde{b}_n(F_{f,\mathbf{0}}) - \tilde{b}_{n-1}(F_{f,\mathbf{0}}) &= \lambda_{f,z_0}^0 - \lambda_{f,z_0}^1 \\ &= (\Gamma_{f,z_0}^1 \cdot V(f))_{\mathbf{0}} - \mu_{\mathbf{0}} \left( f|_{V(z_0)} \right) \\ &= \left[ (\Gamma_{f,\mathbf{z}}^2 \cdot V(h) \cdot V(f))_{\mathbf{0}} - (\Gamma_{f,\mathbf{z}}^2 \cdot V(f) \cdot V(z_0))_{\mathbf{0}} \right] + (\Gamma_{f,\mathbf{z}}^2 \cdot V(z_0, z_1))_{\mathbf{0}}. \end{aligned}$$

As  $(C \cdot V(z_0))_{\mathbf{0}} = \text{mult}_{\mathbf{0}} C$  for all irreducible components  $C$  of  $\Gamma_{f,\mathbf{z}}^2 \cap V(f)$ , the bracketed quantity above is non-negative. The conclusion follows.  $\square$

**Example 4.3.** To illustrate the content of Theorem 4.2, consider the following example. Let  $f = (x^3 + y^2 + z^5)z$  on  $\mathbb{C}^3$ , with coordinate ordering  $(x, y, z)$ . Then, we have  $\Sigma f = V(x^3 + y^2, z)$ , and

$$\Gamma_{f,(x,y)}^2 = V\left(\frac{\partial f}{\partial z}\right) = V(x^3 + y^2 + 6z^5),$$

which we note has an isolated singularity at  $\mathbf{0}$ .

Then,

$$\begin{aligned} V\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) &= V(2yz, x^3 + y^2 + 6z^5) \\ &= V(y, x^3 + 6z^5) + V(z, x^3 + y^2) \end{aligned}$$

so that  $\Gamma_{f,x}^1 = V(y, x^3 + 6z^5)$ , and  $\Lambda_{f,x}^1$  consists of the single component  $C = V(z, x^3 + y^2)$  with  $\overset{\circ}{\mu}_C = 1$ . It is then immediate that

$$\Gamma_{f,x}^1 = V(y) \cdot \Gamma_{f,(x,y)}^2,$$

so that the second hypothesis of Theorem 4.2 is satisfied. For the first hypothesis, we note that

$$\begin{aligned} \Gamma_{f,(x,y)}^2 \cap V(f) &= V(x^3 + y^2 + 6z^5, (x^3 + y^2 + z^5)z) \\ &= V(5z^5, x^3 + y^2 + z^5) \cup V(x^3 + y^2, z) \\ &= V(x^3 + y^2, z) = C. \end{aligned}$$

Clearly,  $C$  is purely 1-dimensional, and is properly intersected by  $V(x)$  at  $\mathbf{0}$ . Finally, we see that

$$(C \cdot V(x))_{\mathbf{0}} = V(x, z, x^3 + y^2)_{\mathbf{0}} = 2 = \text{mult}_{\mathbf{0}} C,$$

so the two hypotheses of Theorem 4.2 are satisfied.

By Proposition 2.3, Theorem 4.2 guarantees that the following inequality holds:

$$\lambda_{f,x}^0 - \lambda_{f,x}^1 \geq \left( \Gamma_{f,(x,y)}^2 \cdot V(x, y) \right)_{\mathbf{0}}.$$

Let us verify this inequality ourselves. We have

$$\begin{aligned} \lambda_{f,x}^0 &= \left( \Gamma_{f,x}^1 \cdot V\left(\frac{\partial f}{\partial x}\right) \right)_{\mathbf{0}} = V(y, x^3 + 6z^5, 3x^2z)_{\mathbf{0}} \\ &= V(y, x^2, z^5)_{\mathbf{0}} + V(y, z, x^3)_{\mathbf{0}} = 13, \end{aligned}$$

and

$$\lambda_{f,x}^1 = (\Lambda_{f,x}^1 \cdot V(x))_{\mathbf{0}} = V(x, z, x^3 + y^2)_{\mathbf{0}} = 2.$$

Finally, we compute

$$\left( \Gamma_{f,(x,y)}^2 \cdot V(x, y) \right)_{\mathbf{0}} = V(x, y, x^3 + y^2 + 6z^5)_{\mathbf{0}} = 5.$$

Putting this all together, we have

$$\lambda_{f,x}^0 - \lambda_{f,x}^1 = 11 \geq 5 = \left( \Gamma_{f,(x,y)}^2 \cdot V(x, y) \right)_{\mathbf{0}},$$

as expected.

**Example 4.4.** We now give an example where the relative polar curve is **not** defined inside  $\Gamma_{f,\mathbf{z}}^2$  by a single equation, and  $\tilde{b}_n(F_{f,\mathbf{0}}) - \tilde{b}_{n-1}(F_{f,\mathbf{0}}) < 0$ .

Let  $f = (z^2 - x^2 - y^2)(z - x)$ , with coordinate ordering  $(x, y, z)$ . Then, we have  $\Sigma f = V(y, z - x)$ , and

$$\Gamma_{f,\mathbf{z}}^2 = V\left(\frac{\partial f}{\partial z}\right) = V(2z(z - x) + (z^2 - x^2 - y^2)).$$

Similarly,

$$V\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = V(y, 3z + x) + 3V(y, z - x),$$

so that  $\Gamma_{f,x}^1 = V(y, 3z + x)$  and  $\mu^\circ = 3$ . It then follows that  $\Gamma_{f,x}^1$  is not defined by a single equation inside  $\Gamma_{f,(x,y)}^2$ .

To see that  $\tilde{b}_2(F_{f,\mathbf{0}}) - \tilde{b}_1(F_{f,\mathbf{0}}) < 0$ , we note that, up to analytic isomorphism,  $f$  is the homogeneous polynomial  $f = (zx - y^2)z$ . Consequently, we need only consider the global Milnor fiber of  $f$ , i.e.,  $F_{f,\mathbf{0}}$  is diffeomorphic to  $f^{-1}(1)$ . Thus,  $F_{f,\mathbf{0}}$  is homotopy equivalent to  $S^1$ , so that  $\tilde{b}_2(F_{f,\mathbf{0}}) = 0$  and  $\tilde{b}_1(F_{f,\mathbf{0}}) = 1$ .

**Corollary 4.5.** *The [Beta Conjecture](#) is true if the set  $\Gamma_{f,\mathbf{z}}^2$  is smooth and transversely intersected by  $V(z_0, z_1)$  at the origin. In particular, the [Beta Conjecture](#) is true for non-reduced plane curve singularities.*

*Proof.* Suppose that the cycle  $\Gamma_{f,\mathbf{z}}^2 = m[V(\mathfrak{p})]$ , where  $\mathfrak{p}$  is prime. Since the set  $\Gamma_{f,\mathbf{z}}^2$  is smooth,  $A := \mathcal{O}_{\mathcal{U},\mathbf{0}}/\mathfrak{p}$  is regular and so, in particular, is a UFD. The image of  $\partial f/\partial z_1$  in  $A$  factors (uniquely), yielding an  $h$  as in hypothesis (2) of Theorem 4.2.

Furthermore, the transversality of  $V(z_0, z_1)$  to  $\Gamma_{f,\mathbf{z}}^2$  at the origin assures us that, by replacing  $z_0$  by a generic linear combination  $az_0 + bz_1$ , we obtain hypothesis (1) of Theorem 4.2.  $\square$

**Example 4.6.** Consider the case where  $f = z^2 + (y^2 - x^3)^2$  on  $\mathbb{C}^3$ , with coordinate ordering  $(x, y, z)$ ; a quick calculation shows that  $\Sigma f = V(z, y^2 - x^3)$ . Then,

$$\Gamma_{f,(x,y)}^2 = V\left(\frac{\partial f}{\partial z}\right) = V(z)$$

is clearly smooth at the origin and transversely intersected at  $\mathbf{0}$  by the line  $V(x, y)$ , so the hypotheses of Corollary 4.5 are satisfied. Again, we want to verify by hand that the inequality

$$\lambda_{f,x}^0 - \lambda_{f,x}^1 \geq \left(\Gamma_{f,(x,y)}^2 \cdot V(x, y)\right)_{\mathbf{0}}$$

holds.

First, we have

$$\lambda_{f,x}^0 = \left(\Gamma_{f,x}^1 \cdot V\left(\frac{\partial f}{\partial x}\right)\right)_{\mathbf{0}} = V(y, z, 2(y^2 - x^3)(-3x^2))_{\mathbf{0}} = V(y, z, x^5)_{\mathbf{0}} = 5,$$

and

$$\lambda_{f,x}^1 = (\Lambda_{f,x}^1 \cdot V(x))_{\mathbf{0}} = V(x, z, y^2 - x^3)_{\mathbf{0}} = V(x, z, y^2)_{\mathbf{0}} = 2.$$

On the other hand, we have  $\left(\Gamma_{f,(x,y)}^2 \cdot V(x, y)\right)_{\mathbf{0}} = V(x, y, z)_{\mathbf{0}} = 1$ , and we see again that the desired inequality holds.

In the case where  $f$  defines non-reduced plane curve singularity, there is a nice explicit formula for  $\beta_f$ , which we will derive in section 5.



## 5. NON-REDUCED PLANE CURVES

By Corollary 4.5, the [Beta Conjecture](#) is true for non-reduced plane curve singularities. However, in that special case, we may calculate  $\beta_f$  explicitly.

Let  $\mathcal{U}$  be an open neighborhood of the origin in  $\mathbb{C}^2$ , with coordinates  $(x, y)$ .

**Proposition 5.1.** *Suppose that  $f$  is of the form  $f = g(x, y)^p h(x, y)$ , where  $g : (\mathcal{U}, \mathbf{0}) \rightarrow (\mathbb{C}, \mathbf{0})$  is irreducible,  $g$  does not divide  $h$ , and  $p > 1$ . Then,*

$$\beta_f = \begin{cases} (p+1)V(g, h)_{\mathbf{0}} + p\mu_{\mathbf{0}}(g) + \mu_{\mathbf{0}}(h) - 1, & \text{if } h(\mathbf{0}) = 0; \text{ and} \\ p\mu_{\mathbf{0}}(g), & \text{if } h(\mathbf{0}) \neq 0. \end{cases}$$

Thus,  $\beta_f = 0$  implies that  $\Sigma f$  is smooth at  $\mathbf{0}$ .

*Proof.* After a possible linear change of coordinates, we may assume that the first coordinate  $x$  satisfies  $\dim_{\mathbf{0}} \Sigma(f|_{V(x)}) = 0$ , so that  $\dim_{\mathbf{0}} V(g, x) = \dim_{\mathbf{0}} V(h, x) = 0$  as well.

As germs of sets at  $\mathbf{0}$ , the critical locus of  $f$  is simply  $V(g)$ . As cycles,

$$\begin{aligned} V\left(\frac{\partial f}{\partial y}\right) &= \Gamma_{f,x}^1 + \Lambda_{f,x}^1 = V\left(phg^{p-1}\frac{\partial g}{\partial y} + g^p\frac{\partial h}{\partial y}\right) \\ &= V\left(ph\frac{\partial g}{\partial y} + g\frac{\partial h}{\partial y}\right) + (p-1)V(g), \end{aligned}$$

so that  $\Gamma_{f,x}^1 = V\left(ph\frac{\partial g}{\partial y} + g\frac{\partial h}{\partial y}\right)$  and  $\Sigma f$  consists of a single component  $C = V(g)$ . It is a quick exercise to show that, for  $g$  irreducible,  $g$  does not divide  $\frac{\partial g}{\partial y}$ , and so the nearby Milnor number is precisely  $\mu_C^{\circ} = (p-1)$  along  $V(g)$ .

Suppose first that  $h(\mathbf{0}) = 0$ .

Then, by Proposition 2.4,

$$\lambda_{f,x}^0 - \lambda_{f,x}^1 = (\Gamma_{f,x}^1 \cdot V(f))_{\mathbf{0}} - \mu_{\mathbf{0}}(f|_{V(x)}).$$

We then expand the terms on the right hand side, as follows:

$$\begin{aligned} (\Gamma_{f,x}^1 \cdot V(f))_{\mathbf{0}} &= p(\Gamma_{f,x}^1 \cdot V(g))_{\mathbf{0}} + (\Gamma_{f,x}^1 \cdot V(h))_{\mathbf{0}} \\ &= pV\left(g, h\frac{\partial g}{\partial y}\right)_{\mathbf{0}} + V\left(h, g\frac{\partial h}{\partial y}\right)_{\mathbf{0}} \\ &= (p+1)V(g, h)_{\mathbf{0}} + pV\left(g, \frac{\partial g}{\partial y}\right)_{\mathbf{0}} + V\left(h, \frac{\partial h}{\partial y}\right)_{\mathbf{0}}. \end{aligned}$$

Since  $\dim_{\mathbf{0}} V(g, x) = 0$  and  $\dim_{\mathbf{0}} V(h, x) = 0$ , the relative polar curves of  $g$  and  $h$  with respect to  $x$  are, respectively,  $\Gamma_{g,x}^1 = V\left(\frac{\partial g}{\partial y}\right)$  and  $\Gamma_{h,x}^1 = V\left(\frac{\partial h}{\partial y}\right)$ . We can therefore apply Teissier's trick to this last equality to obtain

$$\begin{aligned} (\Gamma_{f,x}^1 \cdot V(f))_{\mathbf{0}} &= (p+1)V(g, h)_{\mathbf{0}} + p\left[V\left(\frac{\partial g}{\partial y}, x\right)_{\mathbf{0}} + \mu_{\mathbf{0}}(g)\right] + \left[V\left(\frac{\partial h}{\partial y}, x\right)_{\mathbf{0}} + \mu_{\mathbf{0}}(h)\right] \\ &= (p+1)V(g, h)_{\mathbf{0}} + p\mu_{\mathbf{0}}(g) + pV(g, x)_{\mathbf{0}} + V(h, x)_{\mathbf{0}} - (p+1). \end{aligned}$$

Next, we calculate the Milnor number of the restriction of  $f$  to  $V(x)$ :

$$\mu_{\mathbf{0}}(f|_{V(x)}) = V\left(\frac{\partial f}{\partial y}, x\right)_{\mathbf{0}} = (\Gamma_{f,x}^1 \cdot V(x))_{\mathbf{0}} + (p-1)V(g, x)_{\mathbf{0}}.$$

Substituting these equations back into our initial identity, we obtain the following:

$$\begin{aligned} \lambda_{f,x}^0 - \lambda_{f,x}^1 &= (p+1)V(g,h)_0 + V(g,x)_0 + V(h,x)_0 \\ &\quad + p\mu_0(g) + \mu_0(h) - (\Gamma_{f,x}^1 \cdot V(x))_0 - (p+1). \end{aligned}$$

We now wish to show that  $(\Gamma_{f,x}^1 \cdot V(x))_0 = V(gh,x)_0 - 1$ . To see this, we first recall that

$$(\Gamma_{f,x}^1 \cdot V(x))_0 = \text{mult}_y \left\{ \left( ph \cdot \frac{\partial g}{\partial y} \right)_{|_{V(x)}} + \left( g \cdot \frac{\partial h}{\partial y} \right)_{|_{V(x)}} \right\},$$

where  $g_{|_{V(x)}}$  and  $h_{|_{V(x)}}$  are (convergent) power series in  $y$  with constant coefficients. If the lowest-degree terms in  $y$  of  $\left( ph \cdot \frac{\partial g}{\partial y} \right)_{|_{V(x)}}$  and  $\left( g \cdot \frac{\partial h}{\partial y} \right)_{|_{V(x)}}$  do not cancel each other out, then the  $y$ -multiplicity of their sum is the minimum of their respective  $y$ -multiplicities, both of which equal  $V(gh,x)_0 - 1$ . We must show that no such cancellation can occur. To this end, let  $g_{|_{V(x)}} = \sum_{i \geq n} a_i y^i$  and  $h_{|_{V(x)}} = \sum_{i \geq m} b_i y^i$  be power series representations in  $y$ , where  $n = \text{mult}_y g_{|_{V(x)}}$  and  $m = \text{mult}_y h_{|_{V(x)}}$  (so that  $a_n, b_m \neq 0$ ). Then, a quick computation shows that the lowest-degree term of  $\left( ph \cdot \frac{\partial g}{\partial y} \right)_{|_{V(x)}}$  is  $pn a_n b_m$ , and the lowest-degree term of  $\left( g \cdot \frac{\partial h}{\partial y} \right)_{|_{V(x)}}$  is  $m a_n b_m$ . Consequently, no cancellation occurs, and thus  $(\Gamma_{f,x}^1 \cdot V(x))_0 = V(gh,x)_0 - 1 = n + m - 1$ .

Therefore, we conclude that

$$\beta_f = (p+1)V(g,h)_0 + p\mu_0(g) + \mu_0(h) - 1.$$

Since  $V(g)$  and  $V(h)$  have a non-empty intersection at  $\mathbf{0}$ , the intersection number  $V(g,h)_0$  is greater than one (so that  $\beta_f > 0$ ).

Suppose now that  $h(\mathbf{0}) \neq 0$ . Then, from the above calculations, we find

$$\begin{aligned} (\Gamma_{f,x}^1 \cdot V(f))_0 &= p\mu_0(g) + pV(g,x)_0 - (p+1), \text{ and} \\ \mu_0(f_{|_{V(x)}}) &= pV(g,x)_0 - 1 \end{aligned}$$

so that  $\beta_f = p\mu_0(g)$ .

Recall that, as  $\Sigma f = V(g)$ , the critical locus of  $f$  is smooth at  $\mathbf{0}$  if and only if  $V(g)$  is smooth at  $\mathbf{0}$ ; equivalently, if and only if the Milnor number of  $g$  at  $\mathbf{0}$  vanishes. Hence, when  $\Sigma f$  is *not* smooth at  $\mathbf{0}$ ,  $\mu_0(g) > 0$ , and we find that  $\beta_f > 0$ , as desired.  $\square$

**Remark 5.2.** Suppose that  $f(x,y)$  is of the form  $f = gh$ , where  $g$  and  $h$  are relatively prime, and both have isolated critical points at the origin. Then,  $f$  has an isolated critical point at  $\mathbf{0}$  as well, and the same computation in Proposition 5.1 (for  $\mu_0(f)$  instead of  $\beta_f$ ) yields the formula

$$\mu_0(f) = 2V(g,h)_0 + \mu_0(g) + \mu_0(h) - 1.$$

Thus, the formula for  $\beta_f$  in the non-reduced case collapses to the “expected value” of  $\mu_0(f)$  exactly when  $p = 1$  and  $f$  has an isolated critical point at the origin.

## REFERENCES

- [1] Bobadilla, J. “A Reformulation of Lê’s Conjecture”. In: *Indag. Math.* 17, no. 3 (2006), pp. 345–352.
- [2] “Noeuds, Tresses et Singularités”. In: *Proceedings of the seminar held in Plans-sur-Bex*. Ed. by C. Weber. Monographies de L’Enseignement Mathématique, Mar. 1982.
- [3] Gabrielov, A. M. “Bifurcations, Dynkin Diagrams, and Modality of Isolated Singularities”. In: *Funk. Anal. Pril.* 8 (2) (1974), pp. 7–12.
- [4] Lazzeri, F. *Some Remarks on the Picard-Lefschetz Monodromy*. Quelques journées singulières. Centre de Math. de l’Ecole Polytechnique, Paris, 1974.
- [5] Lê, D. T. “Une application d’un théorème d’A’Campo à l’équisingularité”. In: *Indag. Math.* 35 (1973), pp. 403–409.
- [6] Massey, D. “A New Conjecture, a New Invariant, and a New Non-splitting Result”. In: *ArXiv e-prints* (2014). arXiv:[1410.3316](#) [[math.AG](#)].
- [7] Massey, D. *Lê Cycles and Hypersurface Singularities*. Vol. 1615. Lecture Notes in Math. Springer-Verlag, 1995.
- [8] Massey, D. “The Sebastiani-Thom Isomorphism in the Derived Category”. In: *Compos. Math.* 125 (2001), pp. 353–362.
- [9] Némethi, A. “Generalized local and global Sebastiani-Thom type theorems”. In: *Compositio Math.* 80 (1991), pp. 1–14.
- [10] Némethi, A. “Global Sebastiani-Thom theorem for polynomial maps”. In: *J. Math. Soc. Japan* 43 (1991), pp. 213–218.
- [11] Oka, M. “On the homotopy type of hypersurfaces defined by weighted homogeneous polynomials”. In: *Topology* 12 (1973), pp. 19–32.
- [12] Sakamoto, K. “The Seifert matrices of Milnor fiberings defined by holomorphic functions”. In: *J. Math. Soc. Japan* 26 (4) (1974), pp. 714–721.
- [13] Sebastiani, M. and Thom, R. “Un résultat sur la monodromie”. In: *Invent. Math.* 13 (1971), pp. 90–96.
- [14] Teissier, B. “Cycles évanescents, sections planes et conditions de Whitney”. In: *Astérisque* 7-8 (1973), pp. 285–362.

DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MASSACHUSETTS 02115  
 E-mail address, B. Hepler: [hepler.b@husky.neu.edu](mailto:hepler.b@husky.neu.edu)

DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MASSACHUSETTS 02115  
 E-mail address, D. Massey: [d.massey@neu.edu](mailto:d.massey@neu.edu)